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INVESTIGATION OF VIBRATIONS OF A CYLINDRICAL SHELL CONTAINING A LIQUID

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ABSTRACT: This article presents the results of the theoretical investigations of the nonaxisymmetrical vibrations of a cylindrical shell with a plastic bottom partly filled with a liquid. Each major aspect of the effects of the liquid on the shell is mathematically studied. The differential equations for motion of a shell containing a liquid are derived and conclusions and graphs for different shell conditions are shown. The graphs show that system frequency values decrease with an increase in the level of liquid in the shell. The greatest drop in frequencies is observed at a low fill level up to a certain point after which the frequency drops more slowly.

The axisymmetrical vibrations of shells with an elastic bottom supporting a liquid were studied in [2,5,6]. /27*

This article will present the results of the theoretical investigations of the nonaxisymmetrical vibrations of a cylindrical shell with an elastic bottom partly filled with liquid.

1. Let us examine a cylindrical shell with radius R, length L, and thickness h, partly filled to depth l with an ideal incompressible liquid. The bottom of the shell is assumed to be flat and deformable. The coordinate system shown in Figure 1 was used to solve the problem.

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* Numbers in the right margin indicate pagination in the foreign text.

Let us present the form of the shell and bottom displacements in the following manner, proceeding from references [6, 7] in so doing:

$$u = \cos n\theta \sum_{s=0}^{N} U_s(t) \cos \frac{\lambda_s x}{R};$$

$$v = \sin n\theta \sum_{s=0}^{N} V_s(t) \sin \frac{\lambda_s x}{R}; \qquad (1.1)$$

$$w = \cos n\theta \sum_{s=0}^{N} W_s(t) \sin \frac{\lambda_s x}{R};$$

$$w_1 = \cos n\theta \sum_{i=1}^{M} W_{1i}(t) R_i\left(\frac{\varkappa_{in} r}{R}\right). \qquad (1.2)$$

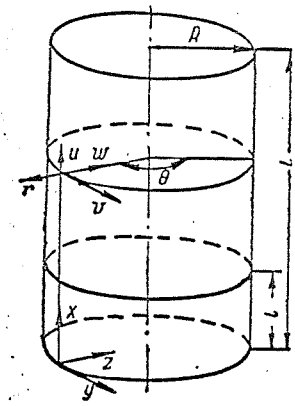

Figure 1.

Here m is the number of axial half-waves; n is the number of equatorial waves; i is the number of nodal circles for the deformable bottom; $\lambda_s = \frac{(m+s)\pi R}{L}$ is a parameter of the axial wave form;

$$R_i\left(\frac{\varkappa_{in} r}{R}\right) = J_n(\varkappa_{in}) I_n\left(\frac{\varkappa_{in} r}{R}\right) - I_n(\varkappa_{in}) J_n\left(\frac{\varkappa_{in} r}{R}\right),$$

where $\varkappa_{in}$ are roots of the transeendental equation /28

$$J_n'(\varkappa_{in}) I_n(\varkappa_{in}) - J_n(\varkappa_{in}) I_n'(\varkappa_{in}) = 0, \qquad (1.3)$$

and J_n, I_n are Bessel functions of the real and imaginary arguments.

Equations (1.1) and (1.2) taken above satisfy the conditions for spherical seating of the shell ends

$$w = \frac{\partial u}{\partial x} = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{when} \quad x = 0, L$$

and the conditions for rigid restraint of the entire edge of the shell bottom

$$w_1 = \frac{\partial w_1}{\partial r} = 0 \quad \text{when} \quad r = R. \qquad (1.4)$$

Deformations of the center surface are associated with the displacements by the equalities [1]

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2; \quad \varepsilon_y = \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2; \qquad (1.5)$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.$$

Let us assume that the displacement of points in the middle surface in a tangential plane does not change shell curvature. Then the equalities expressing the change in curvature are in the form

$$\varkappa_x = -\frac{\partial^2 w}{\partial x^2}; \quad \varkappa_y = -\frac{\partial^2 w}{\partial y^2}; \quad \varkappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}. \qquad (1.6)$$

The stresses attributable to the liquid pressure can be approximated by using the following formulas:

$$\sigma_x^0 = \tau_{xy}^0 = 0; \quad \sigma_y^0 = \frac{\varrho_0 g (l - x)}{h} \quad \text{when } 0 < x < l; \qquad (1.7)$$
$$\sigma_x^0 = \sigma_y^0 = \tau_{xy}^0 = 0 \quad \text{when } l < x < L,$$

where ϱ_o is liquid density; g is the acceleration of gravity.

2. The total potential energy of the system is composed of the potential energies of the shell, U, and bottom V. The magnitude of U can be determined as

$$U = U_1 + U_2 - U_3 \qquad (2.1)$$

Here U_1 and U_2 are the energies of the deformation of the middle surface and of the bending, respectively, and are found through

the following relationship [1]

$$U_1 = \frac{Eh}{2(1-\nu^2)} \int_0^{2\pi R} \int_0^{L} \left(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu \varepsilon_x \varepsilon_y + \frac{1-\nu}{2} \gamma_{xy}^2 \right) dxdy,$$

$$U_2 = \frac{Eh^3}{24(1-\nu^2)} \int_0^{2\pi R} \int_0^{L} [\varkappa_x^2 + \varkappa_y^2 - 2(1-\nu)(\varkappa_x \varkappa_y - \varkappa_{xy}^2)]\, dxdy. \tag{2.2}$$

Let us calculate the work done by forces associated with the internal pressure of the liquid through the approximate formula /29

$$U_3 = -h \int_0^{2\pi R} \int_0^{L} (\sigma_x^0 \varepsilon_x + \sigma_y^0 \varepsilon_y + \tau_{xy}^0 \gamma_{xy})\, dxdy. \tag{2.3}$$

We obtain the following expressions for the shell's potential energy after substituting the relationship at (1.5) - (1.7) in the expressions at (2.2) and (2.3), and taking the equations at (1.1) into consideration

$$U_1 = \frac{\pi EhL}{4R(1-\nu^2)} \sum_{s=0}^{N} \left[\left(\lambda_s^2 + \frac{1-\nu}{2} n^2 \right) U_s^2(t) + \left(n^2 + \frac{1-\nu}{2} \lambda_s^2 \right) V_s^2(t) - \right.$$
$$\left. - (1+\nu)\lambda_s n U_s(t) V_s(t) + W_s^2(t) + 2\nu \lambda_s U_s(t) W_s(t) - 2n V_s(t) W_s(t) \right];$$

$$U_2 = \frac{\pi Eh^3 L}{48R^3(1-\nu^2)} \sum_{s=0}^{N} \left\{ \frac{1-\nu}{2} [nU_s(t) - \lambda_s V_s(t)]^2 + 2\lambda_s n(1-\nu)[nU_s(t) - \right.$$
$$\left. - \lambda_s V_s(t)] W_s(t) + [1 - 2(n^2 + \nu\lambda_s^2) + (\lambda_s^2 + n^2)^2] W_s^2(t) \right\}; \tag{2.4}$$

$$U_3 = -\frac{\pi n^2 \varrho_0 g l^2}{8} \sum_{s=0}^{N} \left\{ \left[\frac{1}{2} - \frac{1 - \cos \frac{2\lambda_s l}{R}}{\left(\frac{2\lambda_s l}{R} \right)^2} \right] W_s(t) + \right.$$
$$\left. + \sum_{\substack{k=0 \\ k \neq s}}^{N} \left[\frac{4\lambda_s \lambda_k}{\frac{l^2}{R^2}(\lambda_s^2 - \lambda_k^2)} + \frac{\cos(\lambda_s + \lambda_k)\frac{l}{R}}{\frac{l^2}{R^2}(\lambda_s + \lambda_k)^2} - \frac{\cos(\lambda_s - \lambda_k)\frac{l}{R}}{\frac{l^2}{R^2}(\lambda_s - \lambda_k)^2} \right] W_k(t) \right\} W_s(t).$$

We note the omission of the squared terms in the formulas at (1.5) in the U_1 calculation because their effect is small compared to that of the linear terms. We will ignore the tensile forces in the plate caused by the liquid in the shell. Then the expression for the potential energy of the bottom of the shell will take the form

$$V=\frac{E^*h^{*3}}{24(1-\nu^2)}\int_0^{2\pi}\int_0^R\left\{\left(\frac{\partial^2\omega_1}{\partial r^2}+\frac{1}{r}\frac{\partial\omega_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\omega_1}{\partial\theta^2}\right)-\right.$$

$$\left.-2(1-\nu)\left[\frac{\partial^2\omega_1}{\partial r^2}\left(\frac{1}{r}\frac{\partial\omega_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\omega_1}{\partial\theta^2}\right)-\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\omega_1}{\partial\theta}\right)\right)^2\right]\right\}rdrd\theta. \qquad (2.5)$$

Substituting the expressions for the deflection of the bottom of the shell at (1.2) in the formula at (2.5), and taking the boundary conditions at (1.4) into consideration, we find

$$V=\frac{\pi E^*h^{*3}}{48R^2(1-\nu^2)}\sum_{i=1}^{M}\left[K_{17}(i)W_{1i}(t)+\sum_{\substack{p=1\\p\neq i}}^{M}K_{18}(p,i)W_{1p}(t)\right]W_{1i}(t), \qquad (2.6)$$

where h^* and E^* are the thickness of and the modulus of elasticity /30 for the bottom, respectively, and $K_{17}(i)$ and $K_{18}(p,i)$ are determined in the following way

$$K_{17}(i)=\varkappa_{in}^4[2J_n^2(\varkappa_{in})I_n^2(\varkappa_{in})-J_n^2(\varkappa_{in})I_{n-1}(\varkappa_{in})I_{n+1}(\varkappa_{in})-$$
$$-I_n^2(\varkappa_{in})J_{n-1}(\varkappa_{in})J_{n+1}(\varkappa_{in})];$$

$$K_{18}(p,i)=2\frac{\varkappa_{in}^2\varkappa_{pn}^2}{\varkappa_{in}^2-\varkappa_{pn}^2}\{\varkappa_{pn}J_n(\varkappa_{in})I_n(\varkappa_{in})[J_n(\varkappa_{pn})I_{n-1}(\varkappa_{pn})+ \qquad (2.7)$$
$$+I_n(\varkappa_{pn})J_{n-1}(\varkappa_{pn})]-\varkappa_{in}I_n(\varkappa_{pn})J_n(\varkappa_{pn})[J_n(\varkappa_{in})I_{n-1}(\varkappa_{in})+$$
$$+I_n(\varkappa_{in})J_{n-1}(\varkappa_{in})]\}.$$

The kinetic energy of the shell T_1, and of the bottom T_3, can be written as follows if the energies of the tangential wave forms are ignored

$$T_1 = \frac{\pi\rho hLR}{4}\sum_{s=0}^{N}\dot{W}_s^2(t);$$

$$T_3 = \frac{\pi\rho^* h^* R^2}{4}\sum_{i=1}^{M}\left[\frac{K_{17}(i)}{\varkappa_{in}^4}\dot{W}_{1i}(t) + \sum_{\substack{p=1\\ p\neq k}}^{M}\frac{K_{18}(i,p)}{\varkappa_{pn}^2\varkappa_{in}^2}\dot{W}_{1p}(t)\right]\dot{W}_{1i}(t). \tag{2.8}$$

3. The velocity potential, φ, for the liquid satisfies the equation of motion

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial x^2} = 0, \tag{3.1}$$

the condition of impermeability for the shell wall and bottom

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = -\frac{\partial w}{\partial t}; \quad \left.\frac{\partial\varphi}{\partial x}\right|_{x=0} = \frac{\partial w_1}{\partial t}, \tag{3.2}$$

the liquid free surface condition [7]

$$\left.\frac{\partial\varphi}{\partial t}\right|_{x=l} = 0$$

and has the following form

$$\varphi_n = \cos n\theta\left\{\sum_{s=0}^{N}\left[\nabla_1(s)\,I_n\left(\frac{\lambda_s r}{R}\right)\sin\frac{\lambda_s x}{R} + \sum_{j=1}^{\infty}\left(\nabla_2(s,j)\,\mathrm{sh}\,\frac{\alpha_{nj}x}{R} + \right.\right.\right.$$

$$\left.\left.\left. + \nabla_3(s,j)\,\mathrm{ch}\,\frac{\alpha_{nj}x}{R}\right)J_n\left(\frac{\alpha_{nj}r}{R}\right)\right]\dot{W}_s(t) + \sum_{i=1}^{M}\sum_{j=1}^{\infty}\left[\nabla_4(i,j)\,\mathrm{sh}\,\frac{\alpha_{nj}x}{R} + \right.\right. \tag{3.3}$$

$$\left.\left. + \nabla_5(i,j)\,\mathrm{ch}\,\frac{\alpha_{nj}x}{R}\right]J_n\left(\frac{\alpha_{nj}r}{R}\right)\dot{W}_{1i}(t)\right\}.$$

The following notations are introduced at this point: /31

$$\nabla_1(s) = -\frac{R}{\lambda_s I_n'(\lambda_s)}; \qquad \nabla_2(s, j) = \frac{2\alpha_{nj}\lambda_s R}{(\alpha_{nj}^2 - n^2) J_n(\alpha_{nj})(\alpha_{nj}^2 + \lambda_s^2)};$$

$$\nabla_3(s, j) = \frac{2\alpha_{nj} R\left(\alpha_{nj}\sin\frac{\lambda_s l}{R} - \lambda_s \operatorname{sh}\frac{\alpha_{nj} l}{R}\right)}{(\alpha_{nj}^2 - n^2)(\alpha_{nj}^2 + \lambda_s^2)\operatorname{ch}\frac{\alpha_{nj} l}{R} J_n(\alpha_{nj})};$$

$$\nabla_4(i, j) = -\frac{4\alpha_{nj}\varkappa_{in} R J_n(\varkappa_{in}) I_n'(\varkappa_{in})}{(\alpha_{nj}^2 - n^2)(\alpha_{nj}^4 - \varkappa_{in}^4) J_n(\alpha_{nj})};$$

$$\nabla_5(i, j) = \frac{4\alpha_{nj}\varkappa_{in} R \operatorname{th}\frac{\alpha_{nj} l}{R} J_n(\varkappa_{in}) I_n'(\varkappa_{in})}{(\alpha_{nj}^2 - n^2)(\alpha_{nj}^4 - \varkappa_{in}^4) J_n(\alpha_{nj})};$$

where α_{nj} is the root of equation $J_n'(\alpha_{nj}) = 0$.

We know [3] that when the condition at (3.2) is taken into consideration the kinetic energy of a liquid can be calculated through the formula

$$T_2 = \frac{\varrho_0}{2}\int_0^l\int_0^{2\pi}\left(\varphi_n\frac{\partial\varphi_n}{\partial r}\right)\Big|_{r=R} R\,dr\,d\theta + \frac{\varrho_0}{2}\int_0^R\int_0^{2\pi}\left(\varphi_n\frac{\partial\varphi_n}{\partial x}\right)\Big|_{x=l} r\,dr\,d\theta - \frac{\varrho_0}{2}\int_0^R\int_0^{2\pi}\left(\varphi_n\frac{\partial\varphi_n}{\partial x}\right)\Big|_{x=0} r\,dr\,d\theta. \tag{3.4}$$

Substituting the expression at (3.3) for the velocity potential of a liquid in the equality at (3.4) and integrating, we find

$$T_2 = \frac{\pi\varrho_0 R^2 l}{4}\left\{\sum_{s=0}^{N}\left[m_1(s)\dot{W}_s^2(t) + \sum_{\substack{k=0\\k\neq s}}^{N} m_2(s, k)\dot{W}_s(t)\dot{W}_k(t)\right] + \sum_{i=1}^{M}\left[m_3(i)\dot{W}_{1i}^2(t) + \sum_{s=0}^{N} m_4(i, s)\dot{W}_s(t)\dot{W}_{1i}(t) + \sum_{\substack{p=1\\p\neq i}}^{M} m_5(i, p)\dot{W}_{1i}(t)\dot{W}_{1p}(t)\right]\right\}. \tag{3.5}$$

where the coefficients of the associated masses of liquid have the /32
form

$$m_1(s) = \frac{I_n(\lambda_s)}{\lambda_s I_n'(\lambda_s)}\left(1 - \frac{\sin\frac{2\lambda_s l}{R}}{\frac{2\lambda_s l}{R}}\right) + \frac{I_n^2(\lambda_s)}{I_n'^2(\lambda_s)}\frac{\sin\frac{2\lambda_s l}{R}}{\frac{2\lambda_s l}{R}} \times$$

$$\times\left[1 - \frac{I_{n-1}(\lambda_s) I_{n+1}(\lambda_s)}{I_n^2(\lambda_s)}\right] - 4\sum_{j=1}^{\infty}\left[2\lambda_s\alpha_{nj}\sin\frac{\lambda_s l}{R} + \left(\alpha_{nj}^2\sin\frac{2\lambda_s l}{R} - \right.\right.$$

$$\left.\left. - \lambda_s^2\right)\operatorname{sh}\frac{\alpha_{nj} l}{R}\right]\left[\frac{\alpha_{nj} l}{R}\left(1 - \frac{n^2}{\alpha_{nj}^2}\right)(\alpha_{nj}^2 + \lambda_s^2)\operatorname{ch}\frac{\alpha_{nj} l}{R}\right]^{-1};$$

$$m_2(s, k) = \frac{I_n(\lambda_s)}{\lambda_s I_n'(\lambda_s)}\left[\frac{\sin(\lambda_s - \lambda_k)\frac{l}{R}}{(\lambda_s - \lambda_k)\frac{l}{R}} - \frac{\sin(\lambda_s + \lambda_k)\frac{l}{R}}{(\lambda_s + \lambda_k)\frac{l}{R}}\right] +$$

$$+ \lambda_s\lambda_k\sin\frac{\lambda_k l}{R}\cos\frac{\lambda_s l}{R}\left[I_{n-1}(\lambda_s) I_{n+1}(\lambda_k) - \right.$$

$$\left. - I_{n+1}(\lambda_s) I_{n-1}(\lambda_k)\right]\left[\frac{l}{R} n(\lambda_k^2 - \lambda_s^2) I_n'(\lambda_s) I_n'(\lambda_k)\right]^{-1} -$$

$$- 4\sum_{j=1}^{\infty}\left\{\left(\lambda_s\sin\frac{\lambda_k l}{R}\cos\frac{\lambda_s l}{R} - \lambda_k\sin\frac{\lambda_s l}{R}\cos\frac{\lambda_k l}{R}\right)\left[\frac{l}{R}\left(1 - \right.\right.\right. \qquad (3.6)$$

$$\left.\left.\left. - \frac{n^2}{\alpha_{nj}^2}\right)(\alpha_{nj}^2 + \lambda_k^2)(\alpha_{nj}^2 + \lambda_s^2)\right]^{-1} + \left[\alpha_{nj}\left(\lambda_s\sin\frac{\lambda_k l}{R} + \lambda_k\sin\frac{\lambda_s l}{R}\right) + \right.\right.$$

$$\left.\left. + \left(\alpha_{nj}^2\sin\frac{\lambda_s l}{R}\sin\frac{\lambda_k l}{R} - \lambda_s\lambda_k\right)\operatorname{sh}\frac{\alpha_{nj} l}{R}\right]\left[\frac{l}{R}\alpha_{nj}^2\left(1 - \frac{n^2}{\alpha_{nj}^2}\right)\left(\alpha_{nj}^2 + \right.\right.\right.$$

$$\left.\left.\left. + \lambda_s^2\right)(\alpha_{nj}^2 + \lambda_k^2)\operatorname{ch}\frac{\alpha_{nj} l}{R}\right]^{-1}\right\};$$

$$m_3(i) = 16\varkappa_{in}^6 J_n^2(\varkappa_{in}) I_n''(\varkappa_{in})\sum_{j=1}^{\infty}\alpha_{nj}\operatorname{th}\frac{\alpha_{nj} l}{R}\left[\frac{l}{R}(\alpha_{nj}^2 - n^2)(\alpha_{nj}^4 - \varkappa_{in}^4)^2\right]^{-1};$$

$$m_4(i, s) = 16\varkappa_{in}^3 J_n(\varkappa_{in}) I_n'(\varkappa_{in})\sum_{j=1}^{\infty}\alpha_{nj}\left(\alpha_{nj}\sin\frac{\lambda_s l}{R} - \right.$$

$$\left. - \lambda_s\operatorname{sh}\frac{\alpha_{nj} l}{R}\right)\left[\frac{l}{R}(\alpha_{nj}^2 - n^2)(\alpha_{nj}^4 - \varkappa_{in}^4)(\alpha_{nj}^2 + \lambda_s^2)\operatorname{ch}\frac{\alpha_{nj} l}{R}\right]^{-1};$$

$$m_5(i, p) = 16\varkappa_{in}^3\varkappa_{pn}^3 J_n(\varkappa_{in}) J_n(\varkappa_{ip}) I_n'(\varkappa_{in}) I_n'(\varkappa_{ip}) \times$$

$$\times\sum_{j=1}^{\infty}\alpha_{nj}\operatorname{th}\frac{\alpha_{nj} l}{R}\left[\frac{l}{R}(\alpha_{nj}^2 - n^2)(\alpha_{nj}^4 - \varkappa_{in}^4)(\alpha_{nj}^4 - \varkappa_{pn}^4)\right]^{-1}.$$

Let us use the approximate correlations obtained in [7] for /33
the coefficients of the associated masses of liquid, m_1, m_2

$$m_1(s) = \frac{I_n(\lambda_s)}{\lambda_s I_n'(\lambda_s)}\left[1 - \frac{\sin\frac{2\lambda_s l}{R}}{\frac{2\lambda_s l}{R}}\right];$$

$$m_2(s,k) = \frac{I_n(\lambda_s)}{\lambda_s I_n'(\lambda_s)}\left[\frac{\sin(\lambda_s-\lambda_k)\frac{l}{R}}{(\lambda_s-\lambda_k)\frac{l}{R}} - \frac{\sin(\lambda_s+\lambda_k)\frac{l}{R}}{(\lambda_s+\lambda_k)\frac{l}{R}}\right]. \qquad (3.7)$$

4. Let us derive the differential equations for the motion of a shell containing a liquid, using Lagrange equations of the second kind, and the expressions for system potential and kinetic energies at (2.4) - (2.8) and (3.5). If the periodic solutions to the equations obtained are found in the form

$$\ddot{U}_s(t) = -\omega^2 U_s(t); \quad \ddot{V}_s(t) = -\omega^2 V_s(t);$$
$$\ddot{W}_s(t) = -\omega^2 W_s(t); \quad \ddot{W}_{1i}(t) = -\omega^2 W_{1i}(t),$$

it develops that the problem reduces to solving a homogeneous system of algebraic equations in terms of the independent parameters of the deflection

$$\begin{aligned}
&a_{11}(s)U_s(t) + a_{12}(s)V_s(t) + a_{13}(s)W_s(t) &&= 0;\\
&a_{21}(s)U_s(t) + a_{22}(s)V_s(t) + a_{23}(s)W_s(t) &&= 0;\\
&a_{31}(s)U_s(t) + a_{32}(s)V_s(t) + a_{33}(s)W_s(t) + \sum_{i=1}^{M}\delta_2(s,i)W_{1i}(t) + \\
&\qquad + \sum_{\substack{k=0\\k\neq s}}^{N}\delta_1(s,k)W_k(t) &&= 0;\\
&\sum_{s=0}^{N}\delta_3(i,s)W_s(t) + \delta_4(i)W_{1i}(t) + \\
&\qquad + \sum_{\substack{p=1\\p\neq i}}^{M}\delta_5(i,p)W_{1p}(t) = 0,
\end{aligned} \qquad (4.1)$$

where the coefficients of the system are the functions m,n,l,ω, and of the geometric dimensions of the shell. We obtain 3N+M+3 equations of motion is we take N+1 terms in the sums at (1.1), and M terms in the sum at (1.2).

Let us find the algebraic equation for determining the vibration frequency spectrum for the system under review by equating to zero the determinant composed of the coefficients of the unknowns.

Let us put N = 0 and M = 1, in the expressions at (1.1) and (1.2). Then let us introduce the following relationship in order to obtain the vibration frequency spectrum for a cylindrical shell with an elastic bottom partly filled with liquid: /34

$$\begin{vmatrix} a_{11}(0) & a_{12}(0) & a_{13}(0) & 0 \\ a_{21}(0) & a_{22}(0) & a_{23}(0) & 0 \\ a_{31}(0) & a_{32}(0) & a_{33}(0) & \delta_2(0,1) \\ 0 & 0 & \delta_3(1,0) & \delta_4(1) \end{vmatrix} = 0. \qquad (4.2)$$

Here

$$a_{11}(0) = \lambda_0^2 + \frac{1-\nu}{2} n^2; \quad a_{12}(0) = a_{21}(0) = -\frac{1+\nu}{2}\lambda_0 n;$$

$$a_{13}(0) = a_{31}(0) = \nu\lambda_0; \quad a_{22}(0) = n^2 + \frac{1-\nu}{2}\lambda_0^2; \quad a_{23}(0) = a_{32}(0) = -n;$$

$$a_{33}(0) = 1 + \beta(\lambda_0^2 + n^2)^2 + \frac{1-\nu^2}{2}\frac{\varrho_0 g R L}{2Eh} n^2 \left(\frac{l}{L}\right)^2 K_2(0) -$$

$$-\Omega^2\left[1 + \frac{\varrho_0 R l}{\varrho h L} m_1(0)\right]; \quad \delta_2(0,1) = -\frac{1}{2}\frac{\varrho_0 R l}{\varrho h L}\Omega^2 m_4(1,0); \qquad (4.3)$$

$$\delta_3(1,0) = -\frac{1}{2}\frac{\varrho_0}{\varrho}\frac{l}{h}\Omega^2 m_4(1,0); \quad \delta_4(1) = -\left[\frac{\varrho^*}{\varrho}\frac{h^*}{h}\frac{K_{17}(1)}{\varkappa_{1n}^4} + \right.$$

$$\left. + \frac{\varrho_0}{\varrho}\frac{l}{h} m_3(1)\right]\Omega^2 + \frac{E^*}{E}\frac{h^{*2}}{12R^2}\frac{h^*}{h} K_{17}(1);$$

$$K_2(0) = 1 - 2\frac{1 - \cos\frac{\lambda_0 l}{R}}{\left(\frac{2\lambda_0 l}{R}\right)^2}; \quad \Omega^2 = \frac{(1-\nu^2)\varrho R^2}{E}\omega^2; \quad \beta = \frac{h^2}{12R^2}.$$

Expanding the determinant at (4.2), and taking the expression at (4.3) for determining the vibration frequency spectrum for the "shell-liquid-elastic bottom" system into consideration, we obtain the equation

$$\Omega^2 = \frac{C_1 + C_2\bar{\Omega}_{mn} \pm \sqrt{(C_1 - C_2\bar{\Omega}^2_{mn})^2 + 4C_1C_2\bar{\Omega}^2_{mn}}}{2(C_2 - C_3)},$$

where

$$C_1 = \frac{E^*}{E}\frac{h^{*2}}{12R^2}\frac{h^*}{h}K_{17}(1); \quad C_2 = \frac{\varrho^*}{\varrho}\frac{h^*}{h}\frac{K_{17}(1)}{\varkappa^4_{1n}} + \frac{\varrho_0}{\varrho}\frac{l}{h}m_3(1); \tag{4.4}$$

$$C_3 = \frac{1}{4}\frac{\left(\frac{\varrho_0}{\varrho}\right)^2\left(\frac{R}{h}\right)^2\left(\frac{l}{L}\right)^2\frac{L}{R}m_4^2(1,0)}{1 + \frac{\varrho_0}{\varrho}\frac{R}{h}\frac{l}{L}m_1(0)}.$$

The magnitudes of $K_{17}(1)$, $m_1(0)$, $m_3(1)$, and $m_4(1,0)$ can be found from the relationships at (2.7), (3.6), and (3.7), and the frequency of the vibrations of a cylindrical shell with a rigid bottom partly filled with liquid is in the form /35

$$\bar{\Omega}^2_{mn} = \left\{\frac{\lambda_0^4(1-\nu^2)}{(\lambda_0^2 + n^2)^2} - \beta\left[2n^2 + 2\nu\lambda_0^2 - (\lambda_0^2 + n^2)^2\right] + \right.$$
$$\left. + \frac{1-\nu^2}{2}\frac{\varrho_0 gRL}{Eh}n^2\left(\frac{l}{L}\right)^2 K_2(0)\right\}\left[1 + \frac{1}{2}\frac{\varrho_0}{\varrho}\frac{R}{h}\frac{R}{L}\frac{1}{\lambda_0^2}\left(2\lambda_0\frac{l}{R} - \right.\right. \tag{4.5}$$
$$\left.\left. - \sin 2\lambda_0\frac{l}{R}\right)\frac{I_n(\lambda_0)}{I_n'(\lambda_0)}\right]^{-1}.$$

The formula at (4.4) was used to investigate frequencies of the vibrations of a cylindrical shell with an elastic bottom supporting a liquid with free surface for different liquid depths.

The calculations show that two subspectra must be distinguished in the spectrum of natural frequencies for a "cylindrical shell-liquid-elastic bottom" system. The frequencies of one of the subspectra are quite close to those of the spectrum for the "shell-liquid-rigid bottom" system. The frequencies of the second subspectrum practically coincide with the frequencies of the spectrum for the "elastic bottom-liquid-rigid shell" system.

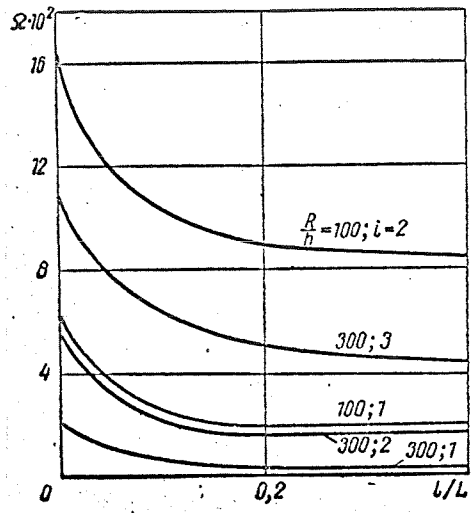

Figure 2.

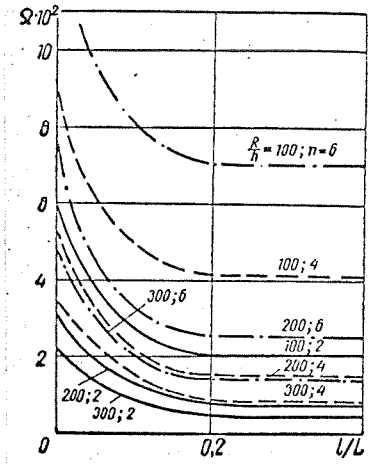

Figure 3.

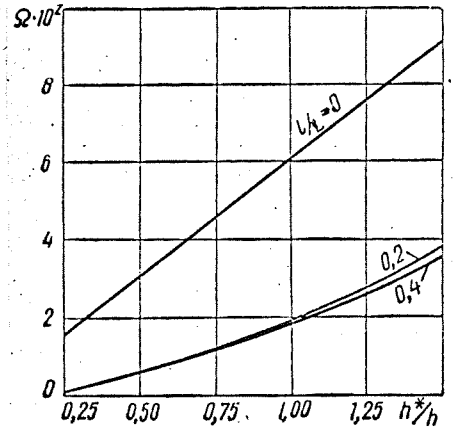

Figure 4.

Figures 2, 3, and 4 show the change in the series of dimensionless frequencies $\Omega \times 10^2$ close to the second subspectrum in terms of the level of liquid in the shell. Dimensionless frequency values were calculated for different relative shell and bottom thicknesses, as well as for a different number of nodal radii, n, and nodal circles, i, for the following parameters

$$\nu = 0{,}3; \quad V = 5 \cdot 10^3 \ м/сек; \quad \frac{R}{L} = 0{,}25; \frac{Q_0}{Q} = 0{,}329032;$$

$$E = 0{,}760275 \cdot 10^{11} \ Н/м^2.$$

/36

The graphs show that system frequency values decrease with increase in the level of liquid in the shell. The greatest drop in frequencies is observed at a low fill level, up to $\ell/L=0.2$, approximately. The frequency drops more slowly when the height of the liquid column rises above this point (see Figures 2 and 3).

An increase in h^*/h, the relative thickness of the bottom, leads to an increase in the magnitude of the series of vibration frequencies for the "rigid shell-liquid-elastic bottom" system, and, the presence of the liquid makes the change in frequency in terms of h^*/h smoother, as will be seen in Figure 4 ($R/h=100$; $i=1$; $n=2$). Consequently the liquid reduces the rate of rise in the dimensionless frequency with increase in h^*/h; that is, it leads to a less pronounced association between the frequency and the bottom thickness.

The calculations show that for $h^*/h=1$ and $n=2$ (see Figure 2), and for $h^*/h=1$ and $i=1$ (see Figure 3), the decrease in the frequencies of the higher wave forms in terms of the height to which the shell is filled with liquid is less intense than in the case of lower wave forms. Actually, when $R/h=100$, $i=1$, and $n=2$, the frequency decreased by a factor of 3.07 for change in ℓ/L from zero to 0.4; when $R/h=100$, $i=1$, and $n=6$ by a factor of 2.09; and when $R/h = 100$, $i = 3$, and $n = 2$ by a factor of 1.55. The obvious explanation is that in the case of vibrations close to axisymmetrical the liquid completes a more intensive vertical movement and, therefore, has a greater effect on change in vibration frequencies.

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